

Geometry of the Grosse-Wulkenhaar Model

Maja Burić ¹ and Michael Wohlgenannt ²

¹*Faculty of Physics*

University of Belgrade, P.O. Box 368

SR-11001 Belgrade, Serbia

Email: majab@phy.bg.ac.yu

²*Institute for Theoretical Physics*

Vienna University of Technology

Wiedner Hauptstr. 8-10

A-1040 Vienna, Austria

Email: michael.wohlgenannt@tuwien.ac.at

ABSTRACT: We analyze properties of a family of finite-matrix spaces obtained by a truncation of the Heisenberg algebra and we show that it has a three-dimensional, noncommutative and curved geometry. Further, we demonstrate that the Heisenberg algebra can be described as a two-dimensional hyperplane embedded in this space. As a consequence of the given construction we show that the Grosse-Wulkenhaar (renormalizable) action can be interpreted as the action for the scalar field on a curved background space. We discuss the generalization to four dimensions.

KEYWORDS: noncommutative geometry, renormalization, noncommutative ϕ^4 model.

Contents

1. Motivation and introduction	1
2. Notation and basic formalism	2
3. The truncated Heisenberg algebra	4
4. Differential geometry in the frame formalism	6
5. Geometry of the truncated Heisenberg space	9
6. Relation to the Grosse-Wulkenhaar action	11
7. Concluding remarks	12

1. Motivation and introduction

Of all noncommutative spaces the Heisenberg algebra, that is the space with constant noncommutativity of coordinates,

$$[x, y] = i\hbar \quad (1.1)$$

has a special role, mainly due to our century-long experience with quantum mechanics. In particular, field theories on (1.1) have been defined in various versions and their classical and quantum properties were analyzed in details. Fields on the Heisenberg algebra are usually represented by functions on \mathbf{R}^n with multiplication given by the Moyal-Weyl product. This representation is mathematically well understood and intuitively appealing; moreover it has an apparent commutative limit $\hbar \rightarrow 0$. Field theories on other algebras, for example on the fuzzy sphere or on the fuzzy CP^n , have also been discussed but certainly not so extensively, [1].

The main advantage of a field theory defined on a space having the structure of Lie algebra with finite-dimensional representations is its finiteness upon quantization: The integral is a trace of a matrix and the functional integration reduces to a well-defined finite expression, namely to an integral over the finite-dimensional space of matrices. This is a feature which one intuitively expects from a theory on a noncommutative space: to regularize divergences. The problem with the Lie-algebra spaces is usually in the definition of a relevant commutative limit, especially if that limit is the flat Minkowski space.

Renormalizability of field theories on the Heisenberg algebra on the other hand is a long discussed issue. If we focus on the scalar field theory defined by action

$$S = \int \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi + \frac{m^2}{2} \varphi^2 + \frac{\lambda}{4!} \varphi^4, \quad (1.2)$$

the calculations done so far converge to the conclusion that the theory is not renormalizable, at least within the usual perturbative schemes, [2]. For a different approach, see [3]. An important exception which was singled out in the last years was found by H. Grosse and R. Wulkenhaar [4, 5]: If we add to the action (1.2) the harmonic oscillator potential term

$$S' = \int \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi + \frac{m^2}{2} \varphi^2 + \frac{\Omega^2}{2} \tilde{x}^\mu \varphi \tilde{x}_\mu \varphi + \frac{\lambda}{4!} \varphi^4, \quad (1.3)$$

the corresponding theory is renormalizable. The physical reason behind this is the additional symmetry which (1.3) possesses called the Langmann-Szabo duality, [6]. This symmetry interchanges the UV and IR sectors of the theory. We shall show that the action (1.3) has another interesting property, namely that the oscillator term can be interpreted geometrically: It is the coupling of the scalar field to the curvature of an appropriately defined noncommutative space.

The plan of the paper is the following: In the initial sections we define and study in some detail the geometry of the truncated Heisenberg space, whose commutation relations are a specific combination of a quantum group and a Lie algebra. We then analyze the properties of its subspace $z = 0$. We show in Section 5 that from the point of view of the algebra and its representations the hyperplane $z = 0$ is a two-dimensional subspace of the truncated Heisenberg algebra. But the dimensionality of the corresponding space of 1-forms is three, and moreover the hyperplane is not flat but curved. Finally, in Section 6 we show that the action for the scalar field coupled to the curvature of the described background space equals the action of the Grosse-Wulkenhaar model. Some additional remarks concerning the properties of the geometry defined by the noncommutative frame formalism are given in the appendices.

2. Notation and basic formalism

Let us introduce the notation. Suppose that the noncommutative space, that is the algebra \mathcal{A} , is generated by hermitian elements, linear operators x^μ ($\mu = 1, \dots, n$) which satisfy the commutation relation

$$[x^\mu, x^\nu] = i\epsilon J^{\mu\nu}(x). \quad (2.1)$$

In the special case of Heisenberg algebra (1.1) we have $J^{\mu\nu} = J^{12} = \text{const}$. Along with coordinate indices μ, ν we shall use frame indices α, β ($= 1, \dots, n$) to denote the components of vectors and 1-forms in the moving frame basis. The constant ϵ which is introduced in (2.1) is a parameter which, similarly to \hbar in (1.1), measures noncommutativity. We shall assume that ϵ is dimensionless; the commutative limit is given by $\epsilon \rightarrow 0$.

If $J^{\mu\nu} = \text{const}$ the noncommutative space can be endowed with a flat connection and a flat metric. The differential d which corresponds to this choice is defined by imposing the following commutation relations on the algebra of 1-forms

$$[x^\mu, dx^\nu] = 0, \quad (2.2)$$

$$[dx^\mu, dx^\nu] = dx^\mu dx^\nu + dx^\nu dx^\mu = 0, \quad (2.3)$$

and the Leibniz rule. Obviously, differential calculus (2.2-2.3) is consistent with the initial algebra. In fact relation (2.2) shows that 1-forms dx^μ can be identified with the elements θ^α of a noncommutative moving frame (vielbein)

$$\theta^\alpha = \delta^\alpha_\mu dx^\mu, \quad (2.4)$$

because property $[x^\mu, \theta^\alpha] = 0$ is sufficient to insure that the frame components of the metric are constant, which means that the space is locally flat*. Derivations $\partial_\mu f$, defined by relation

$$df = \partial_\mu f dx^\mu = e_\alpha f \theta^\alpha, \quad (2.5)$$

are in this case inner, $\partial_\mu f = [p_\mu, f]$, and generated by momenta p_μ ,

$$p_\mu = \frac{1}{i\epsilon}(J^{-1})_{\mu\nu} x^\nu, \quad (2.6)$$

assuming of course that the matrix $J^{\mu\nu}$ is not degenerate. If we calculate the Ricci rotation coefficients $C^\alpha_{\beta\gamma}$ from the definition

$$d\theta^\alpha = -\frac{1}{2}C^\alpha_{\beta\gamma}\theta^\beta\theta^\gamma, \quad (2.7)$$

we see that $C^\alpha_{\beta\gamma}$ vanish for (2.4), that is the space is also globally flat. As one can easily see, momenta p_α can be used to generate the algebra \mathcal{A} instead of coordinates x^μ ; relation (2.6) is a kind of Fourier transformation. The momenta satisfy in the Heisenberg case a commutation relation similar to (1.1):

$$[p_\mu, p_\nu] = -\frac{1}{i\epsilon}(J^{-1})_{\mu\nu} = \text{const.} \quad (2.8)$$

It is in principle possible to define differential structures for arbitrary dependence of the commutator $J^{\mu\nu}(x)$. One has to find a set of 1-forms θ^α which commute with all elements of \mathcal{A} consistently with commutation relations (2.1); this set is then a frame. The choice of the frame might not be unique: the same noncommutative ‘manifold’ can support, in principle, different noncommutative geometries. In the special case of the matrix spaces all derivations are necessarily inner: the definition of the differential d reduces then to the choice of p_α . However the set of p_α is not completely arbitrary. It can be shown that, in order to have relations $d[x^\mu, x^\nu] = i\epsilon dJ^{\mu\nu}$ and $d^2 = 0$ fulfilled, the momenta have to satisfy a quadratic algebra, [7]:

$$[p_\alpha, p_\beta] = \frac{1}{i\epsilon}K_{\alpha\beta} + F^\gamma_{\alpha\beta}p_\gamma - 2i\epsilon Q^{\gamma\delta}_{\alpha\beta}p_\gamma p_\delta, \quad (2.9)$$

where $K_{\alpha\beta}$, $F^\gamma_{\alpha\beta}$ and $Q^{\gamma\delta}_{\alpha\beta}$ are constants or belong to the center of \mathcal{A} . This requirement is an elementary consistency constraint on possible differential structures. As we shall see there are other constraints which make the choice of the differential almost unique.

*The requirement $g^{\alpha\beta} = \text{const}$, that is $[x^\mu, g^{\alpha\beta}] = 0$, is a very stringent one. Here it is, by linearity of the metric, transferred to a less constraining condition $[x^\mu, \theta^\alpha] = 0$.

3. The truncated Heisenberg algebra

Heisenberg algebra (1.1) can be represented in the Fock basis (that is, in the energy representation of the harmonic oscillator) by infinite-dimensional matrices as

$$x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & . & . & . & . \\ 1 & 0 & \sqrt{2} & . & . & . & . \\ 0 & \sqrt{2} & 0 & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & 0 & \sqrt{n-1} & . & . \\ . & . & . & \sqrt{n-1} & 0 & . & . \\ . & . & . & . & . & . & . \end{pmatrix}, \quad (3.1)$$

$$y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 & . & . & . & . \\ 1 & 0 & -\sqrt{2} & . & . & . & . \\ 0 & \sqrt{2} & 0 & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & 0 & -\sqrt{n-1} & . & . \\ . & . & . & \sqrt{n-1} & 0 & . & . \\ . & . & . & . & . & . & . \end{pmatrix}. \quad (3.2)$$

As it is usual in quantum mechanics, x and y are in (3.1-3.2) taken to be dimensionless. Truncation from $\infty \times \infty$ to $n \times n$ matrices consisting of the first n rows and the first n columns,

$$x_n = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & . & . & . & . \\ 1 & 0 & \sqrt{2} & . & . & . & . \\ 0 & \sqrt{2} & 0 & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & 0 & \sqrt{n-1} & . & . \\ . & . & . & \sqrt{n-1} & 0 & . & . \end{pmatrix}, \quad (3.3)$$

$$y_n = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 & . & . & . & . \\ 1 & 0 & -\sqrt{2} & . & . & . & . \\ 0 & \sqrt{2} & 0 & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & 0 & -\sqrt{n-1} & . & . \\ . & . & . & \sqrt{n-1} & 0 & . & . \end{pmatrix}, \quad (3.4)$$

changes the initial algebra $[x, y] = i$ to

$$[x_n, y_n] = i(1 - nP_n), \quad (3.5)$$

where P_n denotes the projector

$$P_n = \begin{pmatrix} 0 & 0 & 0 & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & 0 & 1 \end{pmatrix}. \quad (3.6)$$

The limit $n \rightarrow \infty$ in which (3.5) becomes the Heisenberg algebra is a weak limit; note that it can be written formally as $P_n = 0$, or $nP_n = 0$ as well.

Matrices x_n and y_n have nice geometric interpretation: for fixed n , they describe a finite part of the two-dimensional plane. One can see this from the spectrum of x_n , y_n which consists of all zeroes of the Hermite polynomials H_n , [8], and therefore the expectation values of x_n and y_n are bounded by the largest zero of H_n . When n grows, x_n and y_n approximate larger and larger part of the plane with more and more points, of course not densely. In the limit $n \rightarrow \infty$ one obtains the whole noncommutative x - y plane.

We designate algebra (3.5) the ‘truncated Heisenberg algebra’ because it is obtained by truncation from infinite to finite matrices. In the following we will omit the index n as we shall write it in the form without an explicit n -dependence. We define as usual

$$a = \frac{1}{\sqrt{2}}(x + iy) = \begin{pmatrix} 0 & 1 & 0 & \dots & \dots \\ 0 & 0 & \sqrt{2} & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 & \sqrt{n-1} \\ \dots & \dots & \dots & 0 & 0 \end{pmatrix}, \quad N = a^\dagger a, \quad (3.7)$$

and so on. For fixed n there are additional relations in the algebra, for example

$$a^n = 0, \quad Pa = 0, \quad a^{n-1}(1 - P) = 0, \quad P^2 = P. \quad (3.8)$$

Another well known finite matrix approximation of the Heisenberg algebra was given by Holstein and Primakoff, [9]. For that approximation too the Heisenberg algebra is obtained in the formal limit $n \rightarrow \infty$. From the eigenvalues of coordinate operators one sees that the two given approximations are not unitarily equivalent. The geometry of the Holstein-Primakoff space, given by the differential calculus defined in [7], is also different: it is two-dimensional and flat.

The truncated Heisenberg algebra can be viewed as a three-dimensional noncommutative space generated by coordinates x , y and P . The commutation relation

$$[x, y] = i(1 - nP) \quad (3.9)$$

then has to be completed with the two missing relations. From $Pa = 0$ and its adjoint $a^\dagger P = 0$ we have

$$[x, P] = i(yP + Py), \quad [y, P] = -i(xP + Px). \quad (3.10)$$

In order to think more abstractly we denote $nP = z$ and write the algebra in the form

$$[x, y] = i(1 - z), \quad (3.11)$$

$$[x, z] = i(yz + zy),$$

$$[y, z] = -i(xz + zx).$$

The remaining relations from (3.8) need not to be included in the algebra (3.11), or more precisely in its momentum version (5.2): what is important is that they are stable under differentiation. However the last relation in (3.8) can be used write the algebra in another form: We shall return to this point in more detail in Appendix 1[†]. One should keep in mind that the truncated Heisenberg algebra has finite-dimensional representations for all n . As we shall see, the formal limit $n \rightarrow \infty$ can be consistently viewed as an embedding of the hyperplane $z = 0$ in the given space. Algebra (3.11) is quadratic in its generators: This will allow us to identify the momenta easily and to define a differential calculus.

Before proceeding to that, let us introduce physical dimensions in (3.11). The parameter ϵ in (2.1) is dimensionless and defined in such a way that $\epsilon \rightarrow 0$ gives the commutative limit. In fact we have at least two relevant length scales in the problem. One of them is \sqrt{k} , the scale at which effects of noncommutativity become important. The other scale, as we are presumably dealing with gravity, is the gravitational scale which we denote by μ^{-1} (the Schwarzschild radius or the cosmological constant for example). We assume that $\epsilon = \mu^2 k$. Therefore we write the algebra as

$$[x, y] = i\epsilon\mu^{-2}(1 - \mu z), \quad (3.12)$$

$$[x, z] = i\epsilon(yz + zy),$$

$$[y, z] = -i\epsilon(xz + zx).$$

4. Differential geometry in the frame formalism

Noncommutative differential geometry which we use is defined by a generalization of the moving frame formalism of Cartan. We will briefly review some of its properties; a more detailed exposition can be found in [7]. In the case when momenta p_α generate the space one can express all quantities in terms of them instead as functions of coordinates. This simplifies calculations because momenta obey quadratic relation of a fixed form. The latter can also be written as

$$2P^{\gamma\delta}_{\alpha\beta}p_\gamma p_\delta - F^\gamma_{\alpha\beta}p_\gamma - \frac{1}{i\epsilon}K_{\alpha\beta} = 0. \quad (4.1)$$

Constants $P^{\gamma\delta}_{\alpha\beta}$ define exterior multiplication of 1-forms:

$$\theta^\gamma \theta^\delta = P^{\gamma\delta}_{\alpha\beta} \theta^\alpha \theta^\beta, \quad (4.2)$$

and thus $P^{\gamma\delta}_{\alpha\beta}$ is a projection. In the commutative case $P^{\gamma\delta}_{\alpha\beta}$ is the antisymmetrization,

$$P^{\gamma\delta}_{\alpha\beta} = \frac{1}{2}(\delta^\gamma_\alpha \delta^\delta_\beta - \delta^\gamma_\beta \delta^\delta_\alpha) =: \frac{1}{2}\delta^{\gamma\delta}_{\alpha\beta}, \quad (4.3)$$

[†]We thank the referee for raising this interesting question.

while in the noncommutative case we can write

$$P^{\gamma\delta}{}_{\alpha\beta} = \frac{1}{2}\delta^{\gamma\delta}_{\alpha\beta} + i\epsilon Q^{\gamma\delta}{}_{\alpha\beta}. \quad (4.4)$$

Requirements on hermiticity of the frame forms, hermiticity of the exterior product etc. give additional constraints which we will not discuss here, see [10]. We will however use the fact that coefficients $Q^{\gamma\delta}{}_{\alpha\beta}$ are symmetric in the upper and antisymmetric in the lower pair of indices, evident from (3.11). From the general formalism one can show that the Ricci rotation coefficients (2.7) are linear in momenta and equal to

$$C^{\gamma}{}_{\alpha\beta} = F^{\gamma}{}_{\alpha\beta} - 4i\epsilon Q^{\gamma\delta}{}_{\alpha\beta} p_{\delta}. \quad (4.5)$$

Differential-geometric quantities are defined in complete analogy with the commutative case, which includes for example the requirement of linearity. The (inverse) metric in the frame basis has constant components

$$g^{\alpha\beta} = g(\theta^{\alpha} \otimes \theta^{\beta}) = \text{const.} \quad (4.6)$$

The connection $\omega^{\alpha}{}_{\beta} = \omega^{\alpha}{}_{\gamma\beta}\theta^{\gamma}$ and the torsion $\Theta^{\alpha} = \Theta^{\alpha}{}_{\gamma\beta}\theta^{\gamma}\theta^{\beta}$ are related by the structure equation $\Theta^{\alpha} = d\theta^{\alpha} + \omega^{\alpha}{}_{\beta}\theta^{\beta}$. One can impose additional relations among g , $\omega^{\alpha}{}_{\beta}$ and Θ^{α} : To formulate them it is necessary to introduce a mapping which reverses the order of indices in the tensor product of 1-forms – the ‘flip’ σ :

$$\sigma(\theta^{\gamma} \otimes \theta^{\delta}) = S^{\gamma\delta}{}_{\alpha\beta}\theta^{\alpha} \otimes \theta^{\beta}. \quad (4.7)$$

Coefficients $S^{\gamma\delta}{}_{\alpha\beta}$ are constants and reduce in the commutative limit to $\delta^{\gamma\delta}_{\alpha\beta}$; we write them as

$$S^{\gamma\delta}{}_{\alpha\beta} = \delta^{\gamma\delta}_{\alpha\beta} + i\epsilon T^{\gamma\delta}{}_{\alpha\beta}. \quad (4.8)$$

It seems natural to require that the connection be metric-compatible and that the torsion vanish, as these conditions are rather usual and can always be imposed in commutative geometry. Here they are expressed as

$$\omega^{\alpha}{}_{\beta\gamma}g^{\gamma\delta} + \omega^{\delta}{}_{\gamma\epsilon}S^{\alpha\gamma}{}_{\beta\eta}g^{\eta\epsilon} = 0, \quad (4.9)$$

$$\omega^{\alpha}{}_{\beta\gamma}P^{\beta\gamma}{}_{\delta\epsilon} = \frac{1}{2}C^{\alpha}{}_{\delta\epsilon}, \quad (4.10)$$

respectively. However, (4.9-4.10) is a set of nonlinear algebraic relations in coefficients $g^{\alpha\beta}$, $T^{\alpha\beta}{}_{\gamma\delta}$, $F^{\alpha}{}_{\beta\gamma}$ and $Q^{\alpha\beta}{}_{\gamma\delta}$ and it is not obvious that solutions exist in nontrivial cases. But one can impose these conditions in the commutative limit: for $\epsilon \rightarrow 0$, equations (4.9-4.10) can be linearized and solved, [7]. The solution has the same form as in commutative geometry:

$$\omega_{\alpha\beta\gamma} = \frac{1}{2}(C_{\alpha\beta\gamma} - C_{\beta\gamma\alpha} + C_{\gamma\alpha\beta}), \quad (4.11)$$

or equivalently

$$\omega^{\alpha}{}_{\beta\gamma} = \frac{1}{2}F^{\alpha}{}_{\beta\gamma} + i\epsilon T^{\alpha\delta}{}_{\beta\gamma}p_{\delta}, \quad (4.12)$$

with

$$F^{(\alpha\beta\gamma)} = 0, \quad T^{(\alpha\delta}_{\beta}\gamma) = 0. \quad (4.13)$$

From (4.11) we obtain the expression for the coefficients $T_{\alpha\beta\gamma\delta}$:

$$T_{\alpha\beta\gamma\delta} = 2(-Q_{\alpha\beta\gamma\delta} + Q_{\beta\gamma\delta\alpha} + Q_{\beta\delta\gamma\alpha}); \quad (4.14)$$

in particular,

$$T^{\alpha\beta}_{[\gamma\delta]} = -4Q^{\alpha\beta}_{\gamma\delta}. \quad (4.15)$$

These relations will be used in the calculation of the curvature.

The Riemann curvature is defined by the usual formula

$$\Omega^\alpha_\beta = d\omega^\alpha_\beta + \omega^\alpha_\gamma \omega^\gamma_\beta = \frac{1}{2} R^\alpha_{\beta\rho\sigma} \theta^\rho \theta^\sigma. \quad (4.16)$$

Calculating its coefficients in terms of the momenta we obtain

$$\begin{aligned} R^\alpha_{\beta\rho\sigma} \theta^\rho \theta^\sigma = & 2 \left(T^{\alpha\gamma}_{\sigma\beta} K_{\rho\gamma} - \frac{1}{4} F^\alpha_{\gamma\beta} F^\gamma_{\rho\sigma} + \frac{1}{4} F^\alpha_{\rho\gamma} F^\gamma_{\sigma\beta} \right. \\ & + i\epsilon p_\zeta (F^\zeta_{\rho\gamma} T^{\alpha\gamma}_{\sigma\beta} + F^\alpha_{\gamma\beta} Q^{\gamma\zeta}_{\rho\sigma} - \frac{1}{2} F^\gamma_{\rho\sigma} T^{\alpha\zeta}_{\gamma\beta} + \frac{1}{2} F^\alpha_{\rho\gamma} T^{\gamma\zeta}_{\sigma\beta} + \frac{1}{2} F^\gamma_{\sigma\beta} T^{\alpha\zeta}_{\rho\gamma}) \\ & \left. + (i\epsilon)^2 p_\zeta p_\eta (-2T^{\alpha\gamma}_{\sigma\beta} Q^{\zeta\eta}_{\rho\gamma} + 2T^{\alpha\zeta}_{\gamma\beta} Q^{\gamma\eta}_{\rho\sigma} + T^{\alpha\zeta}_{\rho\gamma} T^{\gamma\eta}_{\sigma\beta}) \right) \theta^\rho \theta^\sigma, \end{aligned} \quad (4.17)$$

that is

$$\begin{aligned} R^\alpha_{\beta\epsilon\delta} = & 2 \left(T^{\alpha\gamma}_{\sigma\beta} K_{\rho\gamma} - \frac{1}{4} F^\alpha_{\gamma\beta} F^\gamma_{\rho\sigma} + \frac{1}{4} F^\alpha_{\rho\gamma} F^\gamma_{\sigma\beta} \right. \\ & + i\epsilon p_\zeta (F^\zeta_{\rho\gamma} T^{\alpha\gamma}_{\sigma\beta} + F^\alpha_{\gamma\beta} Q^{\gamma\zeta}_{\rho\sigma} - \frac{1}{2} F^\gamma_{\rho\sigma} T^{\alpha\zeta}_{\gamma\beta} + \frac{1}{2} F^\alpha_{\rho\gamma} T^{\gamma\zeta}_{\sigma\beta} + \frac{1}{2} F^\gamma_{\sigma\beta} T^{\alpha\zeta}_{\rho\gamma}) \\ & \left. + (i\epsilon)^2 p_\zeta p_\eta (-2T^{\alpha\gamma}_{\sigma\beta} Q^{\zeta\eta}_{\rho\gamma} + 2T^{\alpha\zeta}_{\gamma\beta} Q^{\gamma\eta}_{\rho\sigma} + T^{\alpha\zeta}_{\rho\gamma} T^{\gamma\eta}_{\sigma\beta}) \right) P^{\rho\sigma}_{\delta\epsilon}. \end{aligned} \quad (4.18)$$

The curvature is a second-order polynomial in the momenta. Note that, as momenta are defined by a relation of the type (2.6), $i\epsilon p_\alpha$ is of the same order of magnitude as x^μ even if ϵ is small. We write the curvature as the sum of two terms

$$R^\alpha_{\beta\epsilon\delta} = R_0^\alpha_{\beta\epsilon\delta} + i\epsilon R_1^\alpha_{\beta\epsilon\delta}, \quad (4.19)$$

with

$$\begin{aligned} R_0^\alpha_{\beta\epsilon\delta} = & \left(T^{\alpha\gamma}_{\sigma\beta} K_{\rho\gamma} + \frac{1}{4} F^\alpha_{\rho\gamma} F^\gamma_{\sigma\beta} \right. \\ & + i\epsilon p_\zeta (F^\zeta_{\rho\gamma} T^{\alpha\gamma}_{\sigma\beta} + \frac{1}{2} F^\alpha_{\rho\gamma} T^{\gamma\zeta}_{\sigma\beta} + \frac{1}{2} F^\gamma_{\sigma\beta} T^{\alpha\zeta}_{\rho\gamma}) \\ & \left. + (i\epsilon)^2 p_\zeta p_\eta (-2T^{\alpha\gamma}_{\sigma\beta} Q^{\zeta\eta}_{\rho\gamma} + T^{\alpha\zeta}_{\rho\gamma} T^{\gamma\eta}_{\sigma\beta}) \right) \delta^{\rho\sigma}_{\epsilon\delta} \end{aligned} \quad (4.20)$$

and

$$\begin{aligned}
 R_1^\alpha{}_{\beta\epsilon\delta} = & 2\left(T^{\alpha\gamma}{}_{\sigma\beta}K_{\rho\gamma} + \frac{1}{4}F^\alpha{}_{\rho\gamma}F^\gamma{}_{\sigma\beta}\right. \\
 & + i\epsilon p_\zeta(F^\zeta{}_{\rho\gamma}T^{\alpha\gamma}{}_{\sigma\beta} + \frac{1}{2}F^\alpha{}_{\rho\gamma}T^{\gamma\zeta}{}_{\sigma\beta} + \frac{1}{2}F^\gamma{}_{\sigma\beta}T^{\alpha\zeta}{}_{\rho\gamma}) \\
 & \left. + (i\epsilon)^2 p_\zeta p_\eta(-2T^{\alpha\gamma}{}_{\sigma\beta}Q^{\zeta\eta}{}_{\rho\gamma} + T^{\alpha\zeta}{}_{\rho\gamma}T^{\gamma\eta}{}_{\sigma\beta})\right)Q^{\rho\sigma}{}_{\epsilon\delta}.
 \end{aligned} \tag{4.21}$$

Contracting the Riemann curvature from (4.20-4.21) we obtain for the curvature scalar $R = g^{\beta\delta}R^\alpha{}_{\beta\alpha\delta} = R_0 + i\epsilon R_1$ the following expression:

$$\begin{aligned}
 R_0 = & -8K_{\alpha\gamma}Q^{\alpha\beta}{}_{\beta}{}^\gamma + \frac{1}{4}F^{\alpha\beta\gamma}F_{\alpha\beta\gamma} + 2i\epsilon p_\zeta(-4F^\zeta{}_{\alpha\gamma}Q^{\alpha\beta}{}_{\beta}{}^\gamma - F_{\alpha\beta\gamma}Q^{\zeta\alpha\beta\gamma}) \\
 & + 4(i\epsilon)^2 p_\zeta p_\eta(4Q^{\alpha\beta}{}_{\beta}{}^\gamma Q^{\zeta\eta\alpha\gamma} - 4Q^{\gamma\beta}{}_{\beta}{}^\zeta Q_{\gamma\alpha}{}^{\alpha\eta} - Q^{\zeta\alpha\beta\gamma}Q^\eta{}_{\alpha\beta\gamma} + 2Q^{\zeta\alpha\beta\gamma}Q^\eta{}_{\gamma\alpha\beta}), \\
 R_1 = & 4(i\epsilon)[p_\rho, p_\gamma]T^{\alpha\gamma}{}_{\sigma}{}^\beta Q^{\rho\sigma}{}_{\alpha\beta} - (i\epsilon)^2[p_\zeta, p_\eta]T^{\gamma\zeta\rho\alpha}T_\gamma{}^{\eta\sigma\beta}Q_{\rho\sigma\alpha\beta}.
 \end{aligned} \tag{4.22}$$

The expressions for the Ricci tensor are given in Appendix 2.

5. Geometry of the truncated Heisenberg space

From the discussion of the previous sections we see that geometry is defined by the choice of p_α : Therefore at the first sight it appears to be rather arbitrary. However we have also seen that the additional requirements like metric compatibility or vanishing of the torsion induce additional constraints, for example $F^{(\alpha\beta\gamma)} = 0$. We will in addition impose the condition that in the limit $n \rightarrow \infty$ both algebra and differentials tend to the values which they have in the Heisenberg algebra. This fixes the momenta almost uniquely,

$$\epsilon p_1 = i\mu^2 y, \quad \epsilon p_2 = -i\mu^2 x, \quad \epsilon p_3 = i\mu\left(\mu z - \frac{1}{2}\right). \tag{5.1}$$

The momentum algebra is therefore given by

$$\begin{aligned}
 [p_1, p_2] &= \frac{\mu^2}{2i\epsilon} + \mu p_3, \\
 [p_2, p_3] &= \mu p_1 - i\epsilon(p_1 p_3 + p_3 p_1), \\
 [p_3, p_1] &= \mu p_2 - i\epsilon(p_2 p_3 + p_3 p_2),
 \end{aligned} \tag{5.2}$$

while the nonvanishing structure coefficients have the values

$$K_{12} = \frac{\mu^2}{2}, \quad F^1{}_{23} = \mu, \quad Q^{13}{}_{23} = \frac{1}{2}, \quad Q^{23}{}_{31} = \frac{1}{2}, \tag{5.3}$$

and those obtained by symmetries, for example $Q^{23}{}_{31} = Q^{32}{}_{31} = -Q^{23}{}_{13}$. We shall assume that the truncated Heisenberg space has diagonal metric of Euclidean signature, $(+++)$ and

we will take the connection in the form (4.12). From (4.14) we obtain that the nonvanishing $T_{\alpha\beta\gamma\delta}$ are

$$\begin{aligned} T_{1332} &= 2, & T_{1233} &= 2, & T_{2133} &= -2, \\ T_{2331} &= -2, & T_{3132} &= 2, & T_{3231} &= -2, \end{aligned}$$

so the connection 1-form is given by

$$\begin{aligned} \omega_{12} &= -\omega_{21} = \left(-\frac{\mu}{2} + 2i\epsilon p_3\right)\theta^3 = \mu\left(\frac{1}{2} - 2\mu z\right)\theta^3, \\ \omega_{13} &= -\omega_{31} = \frac{\mu}{2}\theta^2 + 2i\epsilon p_2\theta^3 = \frac{\mu}{2}\theta^2 + 2\mu^2 x\theta^3, \\ \omega_{23} &= -\omega_{32} = -\frac{\mu}{2}\theta^1 - 2i\epsilon p_1\theta^3 = -\frac{\mu}{2}\theta^1 + 2\mu^2 y\theta^3. \end{aligned} \quad (5.4)$$

The scalar curvature, from (4.22), is

$$R = R_0 = \frac{11\mu^2}{2} + 4i\epsilon\mu p_3 - 8(i\epsilon)^2(p_1^2 + p_2^2) = \frac{15\mu^2}{2} - 4\mu^3 z - 8\mu^4(x^2 + y^2). \quad (5.5)$$

It is important to understand exactly the properties of the embedding of the hyperplane $z = 0$ into the truncated Heisenberg algebra. From the point of view of the algebra it is a two-dimensional noncommutative space generated by x and y or by p_1 and p_2 . As on the subspace $p_3 = -\frac{i\mu}{2\epsilon}$ we also have for all functions $e_3 f = [p_3, f] = 0$, consistently with (5.2). However, the cotangent space is three-dimensional. Though from

$$\begin{aligned} dx &= (1 - \mu z)\theta^1 + \mu^2(yz + zy)\theta^3, \\ dy &= (1 - \mu z)\theta^2 - \mu^2(xz + zx)\theta^3, \\ dz &= \mu^2(xz + zx)\theta^1 + \mu^2(yz + zy)\theta^2 \end{aligned} \quad (5.6)$$

for $z = 0$ we obtain $dx = \theta^1$, $dy = \theta^2$, $dz = 0$, it is clear that it is impossible to express one of the basic 1-forms θ^α in terms of the other two, in order to replace it for example in (5.4) or in some other formula.

The fact that the space of 1-forms can have different (higher) dimensionality from the space of coordinates is known in noncommutative geometry. One typical example is the fuzzy sphere [11], where similarly the space ‘itself’ has dimension two whereas the cotangent space is of dimension three. Though for the fuzzy sphere this difference does not show in the calculation of the scalar curvature, it does have important consequences on the construction of gauge theory, [12]. The gauge potential on the fuzzy sphere has the natural dimension three, and all three degrees of freedom are needed to establish the relation with the dynamics of branes on S^3 in string theory for example, [13]. To discuss the commutative limit on the other hand, one has to impose an additional constraint, [14]. In our truncated Heisenberg case we have a three-dimensional cotangent space too.

The value of the connection on $z = 0$ is given by

$$\begin{aligned} \omega_{12} &= -\omega_{21} = \frac{\mu}{2}\theta^3, \\ \omega_{13} &= -\omega_{31} = \frac{\mu}{2}\theta^2 + 2\mu^2 x\theta^3, \\ \omega_{23} &= -\omega_{32} = -\frac{\mu}{2}\theta^1 + 2\mu^2 y\theta^3, \end{aligned} \quad (5.7)$$

and the corresponding value of the scalar curvature is

$$R = \frac{15\mu^2}{2} - 8\mu^4(x^2 + y^2). \quad (5.8)$$

6. Relation to the Grosse-Wulkenhaar action

Having the value of the scalar curvature (5.8) it is not difficult to recognize the relation between the Grosse-Wulkenhaar action (1.3) and the action for the scalar field on a curved space. In the notation of [5], $\tilde{x}_\mu = ip_\mu$ so we have $\tilde{x}^\mu \tilde{x}_\mu = -\frac{\mu^4}{\epsilon^2} x^\mu x_\mu$. Using the cyclicity under the integral we have

$$\int \tilde{x}^\mu \varphi \tilde{x}_\mu \varphi = \int -\frac{1}{2} [p_\mu, \varphi] [p^\mu, \varphi] + \tilde{x}^\mu \tilde{x}_\mu \varphi^2, \quad (6.1)$$

and the Grosse-Wulkenhaar action can be rewritten as

$$S = \int \frac{1}{2} \left(1 - \frac{\Omega^2}{2}\right) \partial_\mu \varphi \partial^\mu \varphi + \frac{m^2}{2} \varphi^2 + \frac{\Omega^2}{2} \tilde{x}^\mu \tilde{x}_\mu \varphi^2 + \frac{\lambda}{4!} \varphi^4. \quad (6.2)$$

On the other hand, the action for the scalar field non-minimally coupled to the curvature is given by

$$S' = \int \sqrt{g} \left(\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi + \frac{M^2}{2} \varphi^2 - \frac{\xi}{2} R \varphi^2 + \frac{\Lambda}{4!} \varphi^4 \right). \quad (6.3)$$

We have seen already that for $z = 0$, $e_\alpha = \delta_\alpha^\mu \partial_\mu$ for $\mu = 1, 2$, $e_3 = 0$ and $\sqrt{g} = 1$. Therefore we find that (6.2) and (6.3) are the same up to an overall rescaling

$$S = \kappa S', \quad (6.4)$$

if we identify

$$1 - \frac{\Omega^2}{2} = \kappa, \quad m^2 = \kappa(M^2 - \xi a), \quad \frac{\Omega^2 \mu^4}{\epsilon^2} = \kappa \xi b, \quad \lambda = \kappa \Lambda, \quad (6.5)$$

and a and b from (5.8), $a = \frac{15\mu^2}{2}$, $b = 8\mu^4$.

The constant part of the curvature renormalizes the mass of the scalar field, while the space-dependent part gives the harmonic oscillator potential. The coupling constant ξ is not a priori fixed but can be related to Ω . If we identify the two actions at the self-duality point $\Omega = 1$ we obtain

$$\xi = \frac{\Omega^2 \mu^4}{\epsilon^2 \kappa b} = \frac{1}{4\epsilon^2}. \quad (6.6)$$

Note that if $M = 0$ in the initial action (6.3) our mechanism induces a mass term with negative sign and thus we obtain a Lagrangian containing the Higgs potential and spontaneous symmetry breaking.

It is not difficult to generalize the given construction from two to four spatial dimensions and reach the same conclusion for the four-dimensional Grosse-Wulkenhaar model.

The key element is that there exist a set of coordinates in which noncommutativity $J^{\mu\nu}$ has the canonical, block-diagonal form

$$J^{\mu\nu} \sim \begin{pmatrix} & 1 & \\ -1 & & \\ & & 1 \\ & & -1 \end{pmatrix}. \quad (6.7)$$

In these coordinates the four-dimensional noncommutative space \mathcal{A} is a direct product of two two-dimensional spaces, $\mathcal{A} = \mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)}$; $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ commute. Then, finite-matrix approximations to \mathcal{A} can be defined by taking direct products of the approximations to $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ which are described above. The four-dimensional \mathcal{A} is a subspace of the six-dimensional space defined by $z^{(1)} = 0$, $z^{(2)} = 0$. Clearly, as the product spaces commute, the scalar curvature is the sum of curvatures

$$R = R^{(1)} + R^{(2)}; \quad (6.8)$$

similar holds true for the Laplacian. Thus one can make the same identification of the constants (6.5) and of the actions in four dimensions as one does in two.

The procedure to obtain the four-dimensional action described above is the simplest one can think of. It apparently breaks the symmetry among x^μ : but in fact this symmetry is broken from the start by the values of the components of $J^{\mu\nu}$. It would be interesting to find another, minimal in the sense of dimensionality and finite, approximation to the four-dimensional Heisenberg algebra.

7. Concluding remarks

To summarize: We have shown that it is possible, through a sequence of matrix representations, to define a noncommutative space which has the same algebra but a different geometry from the Moyal-deformed space. Specific properties of the described space allow the interpretation of the oscillator term in the Grosse-Wulkenhaar model as a coupling to the background curvature. In particular, the given picture explains the absence of the translation invariance in the Grosse-Wulkenhaar action: The underlying curved space does not possess it. The performed construction can be extended easily to any even-dimensional space.

An important technical detail was the difference in dimensionality between the basic space and the cotangent space. This possibility, typical for noncommutative geometry, has been studied before for the fuzzy sphere, [11, 12, 13, 14]. Its characteristic consequences appear whenever one deals with fields which ‘live’ in the cotangent space, for example gauge fields or linear connection, and are in some ways similar to the Kaluza-Klein reduction. Another fact important to stress is that the set of intermediate matrix spaces discussed above was not introduced arbitrarily; rather, the matrix base was used to establish the first proof of renormalizability of the model [4], relying on estimating the decay properties of the propagator.

An interesting point is that we have identified a model in which the field theory renormalization is (or can be interpreted to be) done effectively by the curvature. The old idea of Pauli, Deser and others [15] that gravity can regularize field theory is here realized in a very specific way, in the setting of noncommutative geometry. However it does not directly correspond to the common intuition that regularization works through the uncertainty relations; the regularization is rather indirect, through the curvature, [16]. There is of course another ingredient of the given construction which is hard to disentangle from its geometric aspects: the finiteness of the representation. This element might be even primary in considering renormalizability, and perhaps indicates an advantage of the theories which can be regularized through matrix models.

The model described in the paper opens, in our opinion, interesting new possibilities to understand relations between noncommutative gravity and noncommutative field theory. One possibility to interpret the oscillator term, given previously in [6], is to relate it to the coupling of the complex scalar to the external magnetic field. Here it is the external gravitational field which couples to the scalar φ . (In our approach both real and complex scalar fields have the same behavior, as their coupling to gravity is of the same form.) If the geometric interpretation has a deeper physical meaning, it should provide also a description for the other fields, for example gauge fields or spinors. Given the fixed background geometry the corresponding actions should be straightforward to define; we will analyze properties of such models in our future work. Another important aspect which should be addressed in the future is to understand what is exactly the role of the of the Langmann-Szabo duality in the given framework and in particular, whether there is a relation to the frame formalism.

Acknowledgment The authors would like to thank J. Madore and H. Grosse for exciting discussions during the work on this manuscript. This work of was supported in part by ESF grant 2747 through the Quantum Gravity Network and by 141036 grant of MNTR, Serbia.

Appendix 1

We mentioned briefly in Section 3 that the relations (3.8) need not be included in the algebra of coordinates (3.12), or of the momenta (5.2). The reason is, that these relations are not expressed in the form of commutators – while the differential calculus is, $df = [p_\alpha, f]\theta^\alpha$. On the other hand, (3.8) are consistent that is stable under differentiation, for example $d(Pa) = 0$, $d(P^2 - P) = 0$ etc.; this can be checked easily.

However the projector condition from (3.8) can be used to modify the algebra (3.12) and to write it in another, also quadratic, form. (The quadratic form is preferred because then the identification of momenta is much easier.) Using $n\mu z = (\mu z)^2$ we can rewrite

(3.12) as

$$\begin{aligned} [x, y] &= i\epsilon\mu^{-2}\left(1 - \frac{\mu^2 z^2}{n}\right), \\ [x, z] &= i\epsilon(yz + zy), \\ [y, z] &= -i\epsilon(xz + zx). \end{aligned} \quad (7.1)$$

The linear terms in (7.1) are absent, so it is simpler and perhaps more natural to choose the momenta as:

$$\epsilon p'_1 = i\mu^2 y, \quad \epsilon p'_2 = -i\mu^2 x, \quad \epsilon p'_3 = i\mu^2 z. \quad (7.2)$$

Of course, this changes the differential calculus but not much as we shall shortly see. We are interested in the curvature. The momentum algebra is now given by

$$\begin{aligned} [p'_1, p'_2] &= \frac{\mu^2}{i\epsilon} - \frac{i\epsilon}{n}(p'_3)^2, \\ [p'_2, p'_3] &= -i\epsilon(p'_1 p'_3 + p'_3 p'_1), \\ [p'_3, p'_1] &= -i\epsilon(p'_2 p'_3 + p'_3 p'_2), \end{aligned} \quad (7.3)$$

and the corresponding nonvanishing structure coefficients are

$$K'_{12} = \mu^2, \quad Q'^{13}_{23} = \frac{1}{2}, \quad Q'^{23}_{31} = \frac{1}{2}, \quad Q'^{33}_{12} = \frac{1}{2n}. \quad (7.4)$$

Calculating the scalar curvature from the connection defined in the same manner as before, we obtain

$$\begin{aligned} R' &= 8\mu^2 - 8(i\epsilon)^2((p'_1)^2 + (p'_2)^2) + \frac{4(i\epsilon)^3}{n}[p'_1, p'_2] \\ &= 8\mu^2 - 8\mu^4(x^2 + y^2) - \frac{4\epsilon^2}{n}\mu^2 + \frac{4\epsilon^2}{n^2}\mu^4 z^2. \end{aligned} \quad (7.5)$$

On the subspace $z = 0$, $n \rightarrow \infty$ it reduces to

$$R' = 8\mu^2 - 8\mu^4(x^2 + y^2). \quad (7.6)$$

The result is rather interesting: the desired quadratic dependence on coordinates appears again, though the value of the scalar curvature is not exactly the same. This change one can attribute to the change of the momenta (7.2), that is to the change of the differential d . It might be interesting to compare in some detail the respective connections ω and ω' . Invariance and properties of the geometric characteristics of noncommutative spaces under the change of generators of the algebra are certainly an important topic which deserves further study.

Appendix 2

The components of the Ricci curvature which correspond to the Riemann tensor (4.18) are given by

$$\begin{aligned}
 2R_{0\beta\sigma} &= 2R_0{}^\alpha{}_{\beta\alpha\sigma} = T^{\alpha\gamma}{}_{\sigma\beta}K_{\alpha\gamma} - T^{\alpha\gamma}{}_{\alpha\beta}K_{\sigma\gamma} - \frac{1}{4}F^\alpha{}_{\gamma\beta}F^\gamma{}_{\alpha\sigma} \\
 &+ i\epsilon p_\zeta \left(F^\zeta{}_{\delta\gamma}T^{\alpha\gamma}{}_{\sigma\beta} - F^\zeta{}_{\sigma\gamma}T^{\alpha\gamma}{}_{\alpha\beta} - F^\gamma{}_{\alpha\sigma}T^{\alpha\zeta}{}_{\gamma\beta} - \frac{1}{2}F^\alpha{}_{\sigma\gamma}T^{\gamma\zeta}{}_{\alpha\beta} \right. \\
 &\quad \left. - \frac{1}{2}F^\alpha{}_{\gamma\beta}T^{\gamma\zeta}{}_{\alpha\sigma} + \frac{1}{2}F^\alpha{}_{\gamma\beta}T^{\gamma\zeta}{}_{\sigma\alpha} - \frac{1}{2}F^\gamma{}_{\alpha\beta}T^{\alpha\zeta}{}_{\sigma\gamma} + \frac{1}{2}F^\gamma{}_{\sigma\beta}T^{\alpha\zeta}{}_{\alpha\gamma} \right) \\
 &+ (i\epsilon)^2 p_\zeta p_\eta \left(-T^{\alpha\zeta}{}_{\gamma\beta}T^{\gamma\eta}{}_{\alpha\sigma} + T^{\alpha\zeta}{}_{\gamma\beta}T^{\gamma\eta}{}_{\sigma\alpha} + T^{\alpha\zeta}{}_{\alpha\gamma}T^{\gamma\eta}{}_{\sigma\beta} - T^{\alpha\zeta}{}_{\sigma\gamma}T^{\gamma\eta}{}_{\alpha\beta} \right. \\
 &\quad \left. + \frac{1}{2}T^{\alpha\gamma}{}_{\sigma\beta}T^{\zeta\eta}{}_{\alpha\gamma} - \frac{1}{2}T^{\alpha\gamma}{}_{\sigma\beta}T^{\zeta\eta}{}_{\gamma\alpha} - \frac{1}{2}T^{\alpha\gamma}{}_{\alpha\beta}T^{\zeta\eta}{}_{\sigma\gamma} + \frac{1}{2}T^{\alpha\gamma}{}_{\alpha\beta}T^{\zeta\eta}{}_{\gamma\sigma} \right), \\
 R_{1\beta\nu} &= \left(T^{\alpha\gamma}{}_{\sigma\beta}K_{\rho\gamma} + \frac{1}{4}F^\alpha{}_{\rho\gamma}F^\gamma{}_{\sigma\beta} \right. \\
 &\quad \left. + i\epsilon p_\zeta (F^\zeta{}_{\rho\gamma}T^{\alpha\gamma}{}_{\sigma\beta} + \frac{1}{2}F^\alpha{}_{\rho\gamma}T^{\gamma\zeta}{}_{\sigma\beta} + \frac{1}{2}F^\gamma{}_{\sigma\beta}T^{\alpha\zeta}{}_{\rho\gamma}) \right. \\
 &\quad \left. + (i\epsilon)^2 p_\zeta p_\eta (-2T^{\alpha\gamma}{}_{\sigma\beta}Q^{\zeta\eta}{}_{\rho\gamma} + T^{\alpha\zeta}{}_{\rho\gamma}T^{\gamma\eta}{}_{\sigma\beta}) \right) Q^{\rho\sigma}{}_{\alpha\nu}.
 \end{aligned}$$

In principle, they enter the field actions when gravity couples to the other fields. On the subspace $z = 0$ of the truncated Heisenberg space the value of the Ricci tensor is

$$R_{\alpha\beta} = \begin{pmatrix} \frac{3\mu^2}{2} - 4\mu^4 x^2 & -2\mu^4(xy + yx) + i\frac{\epsilon\mu^2}{4} & 2\mu^3 y + 2i\epsilon\mu^3 x \\ -2\mu^4(xy + yx) - i\frac{\epsilon\mu^2}{4} & \frac{3\mu^2}{2} - 4\mu^4 y^2 & -2\mu^3 x + 2i\epsilon\mu^3 y \\ 2\mu^3 y - 2i\epsilon\mu^3 x & -2\mu^3 x - 2i\epsilon\mu^3 y & \frac{9\mu^2}{2} - 4\mu^4(x^2 + y^2) \end{pmatrix}.$$

References

- [1] J. Medina and D. O'Connor, JHEP **0311**, 051 (2003) [arXiv:hep-th/0212170], H. Grosse and H. Steinacker, Nucl. Phys. B **707**, 145 (2005) [arXiv:hep-th/0407089], W. Behr, F. Meyer and H. Steinacker, JHEP **0507**, 040 (2005) [arXiv:hep-th/0503041].
- [2] I. Chepelev and R. Roiban, JHEP **0103**, 001 (2001) [arXiv:hep-th/0008090], I. Chepelev and R. Roiban, JHEP **0005**, 037 (2000) [arXiv:hep-th/9911098].
- [3] S. Cho, R. Hinterding, J. Madore and H. Steinacker, Int. J. Mod. Phys. D **9** (2000) 161 [arXiv:hep-th/9903239].
- [4] H. Grosse and R. Wulkenhaar, JHEP **0312**, 019 (2003) [arXiv:hep-th/0307017].
- [5] H. Grosse and R. Wulkenhaar, Commun. Math. Phys. **256**, 305 (2005) [arXiv:hep-th/0401128], H. Grosse and R. Wulkenhaar, Lett. Math. Phys. **71** (2005) 13.
- [6] E. Langmann and R. J. Szabo, Phys. Lett. B **533** (2002) 168 [arXiv:hep-th/0202039].

- [7] J. Madore, “An Introduction To Noncommutative Differential Geometry And Its Physical Applications,” Lond. Math. Soc. Lect. Note Ser. **257** (2000) 1.
- [8] A. P. Polychronakos, arXiv:0706.1095 [hep-th].
- [9] T. Holstein and H. Primakoff, Phys. Rev. **58** (1940) 1098.
- [10] G. Fiore and J. Madore, [arXiv:math/9806071].
- [11] J. Madore, Class. Quant. Grav. **9** (1992) 69.
- [12] H. Grosse and J. Madore, Phys. Lett. B **283** (1992) 218.
- [13] A. Y. Alekseev, A. Recknagel and V. Schomerus, JHEP **0005**, 010 (2000) [arXiv:hep-th/0003187].
- [14] D. Karabali, V. P. Nair and A. P. Polychronakos, Nucl. Phys. B **627** (2002) 565
H. Steinacker, Nucl. Phys. B **679** (2004) 66 [arXiv:hep-th/0307075] D. O’Connor and B. Ydri, JHEP **0611** (2006) 016 [arXiv:hep-lat/0606013].
- [15] S. Deser, Rev. Mod. Phys. **29**, 417 (1957), C. J. Isham, A. Salam and J. A. Strathdee, Phys. Rev. D **3**, 1805 (1971).
- [16] J. Madore and J. Mourad, J. Math. Phys. **39**, 423 (1998) [arXiv:gr-qc/9607060].